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Dead-time effects in photon counting statistics

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Abstract. The paper deals with the modifications in photocount distributions due to dead-time following every registered count. In order to study the effects due to dead-time, the distribution of the interval between successive photocounts is studied. For the case of a constant dead-time, the mean count rate is derived using the dead-time neglected two-fold counting generating function. The result is also generalised to cover the case when the dead-time has a distribution with the different dead-times independently and identically distributed. In view of the complexity of the problem, a lower bound is obtained for the mean square number of photocounts in an arbitrary interval.

1. Introduction

The analysis of photocount distribution has, in recent years, gained practical importance mainly due to the recognition of the situation that the correlational properties of photon beams can be brought out by the characteristics of the photoelectric counts. Several surveys are now available (Mandel 1963, Mandel and Wolf 1965, Pike 1969, Mehta 1970, Cummins and Pike 1974, Chiu 1974, Troup and Turner 1974) that describe the state of the art in the subject. The usual method of arriving at the photocount distribution, due to Mandel (1963), consists in observing that the photocounts are governed by a Poisson distribution with parameter $\alpha E(T)$ where α is the photo efficiency of the detector and

$$E(T)=\int_{t}^{t+T} I(t')\,\mathrm{d}t' \tag{1.1}$$

where T is large compared to the coherence time of the incident beam and $I(t)$ is the intensity of the incident beam. The photocount distribution is arrived at by making an ensemble average of the Poisson distribution over E. This method has been improvised and extended over the years to cover the most general case when the time interval T is arbitrary. Jakeman and Pike (1969) provided an extensive table containing information regarding the state of knowledge of photocount statistics of Gaussian light up to 1969. The state of the art was further improved by Srinivasan and Sukavanam (1971, 1972) who provided a method of arriving at explicit formulae for the photocount distribution when the spectral profile is meromorphic in character. A further improvement to cover analytic profiles was carried out by Srinivasan (1974a) (see also Srinivasan and Sukavanam 1977). Meanwhile there were parallel attempts to characterise the light beam characteristics by higher-order photocount statistics. Dialetis (1969), Cantrell (1971) and Srinivasan *et al* (1973) have derived explicit formulae for many-time photocount distributions for beams with fairly general spectral profiles.

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While these developments bring the study of photon counting statistics to a fairly good degree of saturation, there is a small gap due to the modification of the photocount statistics arising from the presence of dead-time effects. The inclusion of dead-time effects in its full generality is fraught with difficulties almost intractable in nature. Consequently resort is generally made to approximate formulae for the distribution of the number of photoelectrons detected in any interval of time. The first results in this direction were obtained by de Lotto *et al* (1964) who gave corrections to the Poisson emission probability of photoelectrons due to an integrated intensity E falling on the photocathode of a detector when the dead-time a is small compared to the time interval in question. This was further improved by Bédard (1967) who found an explicit expression for the probability mass function of the number of counts in $(0, T)$ where T is small compared to the coherence time. A method of improving the result was indicated in a short communication (Srinivasan 1975) for a detector with a deterministic resolving time. In this paper it is proposed to provide full details of the derivation of the formulae and extend the results to the case when the dead-time has a distribution.

2. Interval distribution of the successive counts for fixed dead-time

We first note that the photocounts constitute a stationary stochastic point process (see Srinivasan 1974b, Macchi 1975) on the time axis by virtue of the optical field being a stationary Gaussian random process. If $V(t)$ is the analytic signal, then corresponding to a specified sample function $V_s(t)$, $p(n, T)$ the probability that the number of photocounts is n in an arbitrary interval (t_0, t_0+T) is given by (see Mandel 1963)

$$p(n, T)=\left(\alpha \int_{t_0}^{t_0+T} I_s(t)\,dt\right)^n\left[\exp\left(-\alpha \int_{t_0}^{t_0+T} I_s(t)\,dt\right)\right](n!)^{-1} \tag{2.1}$$

where

$$I_s(t)=V_s^*(t)V_s(t). \tag{2.2}$$

Since $V_s(t)$ is only a sample function of the optical field $V(t)$, an ensemble average over the right-hand side of (2.1) leads to the final distribution of the number of counts. Such a counting process is known as a doubly stochastic Poisson process or conditioned Poisson process. If at this stage we introduce a dead-time a where a is a fully determinate quantity, no such simple formula like (2.1) for its ensemble average is feasible for the most general case when no restriction is placed on the time interval T. If however T is small compared to the coherence time, then the variation of the optical field $V(t)$ is not great during the interval under consideration and the normal tools of stochastic processes can be used to modify (2.1) appropriately (see Bédard 1967 for details). To deal with the general case, we confine our attention to the stationary distribution of the time intervals between two successive counts. In other words we are interested in the function $f_a(x)$ where

$$f_a(x)=\lim_{\Delta,\Delta'\to 0} \Pr\{N_a(t_0+x+\Delta)-N_a(t_0+x)$$

$$=1>N_a(t_0+x)-N_a(t_0)|N_a(t_0)>N_a(t_0-\Delta')\}/\Delta \tag{2.3}$$

where $N_a(\cdot)$ is the counting process corresponding to the photoelectrons registered by the photodetector. The numerator of the right-hand side of (2.3) denotes the probability that there is no photocount in (t_0, t_0+x) and one photocount in $(t_0+x, t_0+x+$

Δ) conditional upon there being a photocount in the infinitesimal interval $(t_0-\Delta', t_0)$ preceding t_0. This probability when divided by Δ tends to a limit as Δ and Δ' tend to zero, the limit being independent of t_0 by virtue of the stationariness of the process $\{V(t)\}$. Thus the right-hand side of (2.3) is the stationary probability density function (PDF) of the interval between two successive photocounts.

To evaluate the conditional probability on the right-hand side of (2.3) we note that the detector is 'free' to detect the photon incident on it provided the dead-time is not in operation. Let $\beta(t)$ be the probability that the detector is free at any arbitrary time t. Then the probability that the detector gives rise to a photocount† is $\beta(t)\langle I(t)\rangle$. Using the stationary nature of the field we find this to be $\beta\langle I\rangle$ where β is the stationary value of $\beta(t)$. Thus we are led to the conclusion that the average rate of photons that are detected is $\beta\langle I\rangle$. We now demonstrate how β can be determined. From the theory of stationary point processes (see McFadden 1962, Srinivasan 1974b) we have

$$1/\beta\langle I\rangle = \int_0^\infty x f_a(x)\,\mathrm{d}x. \tag{2.4}$$

Thus if $f_a(\cdot)$ is determined, some light can be shed on the photon counting statistics. To determine $f_a(\cdot)$ we use the stationary and doubly stochastic Poisson character of the counting process $N_a(\cdot)$. If $I_s(\cdot)$ is a sample function of the intensity, the counting process is an inhomogeneous Poisson process in the absence of dead-time. However, if dead-times are taken into account, the probability density of the counts is zero in an interval of length a immediately following a count, the other characteristics of the distribution remaining the same. Thus for a given sample function $I_s(\cdot)$, the joint probability density of the first count (measured from t_0) at the point $t_0+x\,(x>a)$ and a count in an infinitesimal interval Δ preceding the point t_0 is given by

$$\Delta\beta I_s(t_0)\left[\exp\left(-\int_{t_0+a}^{t_0+x} I_s(u)\,\mathrm{d}u\right)\right] I_s(t_0+x) \qquad x>a.$$

If an ensemble average is taken over all paths $I_s(\cdot)$ the above expression reduces to

$$\Delta\left\langle \beta I_s(t_0)\left[\exp\left(-\int_{t_0}^{t_0+a} I_s(u)\,\mathrm{d}u\right)\right] I_s(t_0+x)\right\rangle.$$

To obtain $f_a(\cdot)$, we note that $f_a(\cdot)$ is the stationary conditioned probability density of the time to the first photocount measured from t_0 conditioned upon a photocount in an infinitesimal interval preceding t_0. Thus if the constraint of stationariness is imposed and the above expression is divided by the stationary probability that a photocount is recorded in the infinitesimal interval $(t_0-\Delta, t_0)$, we obtain

$$f_a(x)=\begin{cases}\dfrac{\langle\beta I(t_0)[\exp-(\int_{t_0+a}^{t_0+x} I(u)\,\mathrm{d}u)]I(t_0+x)\rangle}{\langle\beta I(t_0)\rangle} & x\geqslant a\\ 0 & x<a\end{cases} \tag{2.5}$$

We note that in the right-hand side of (2.5) β being a positive definite constant gets cancelled and $f_a(\cdot)$ is expressed purely in terms of the random function $I(t)$ or $V(t)$. In other words the statistical characteristics of the counting process $N_a(\cdot)$ (censored process) are expressed in terms of the counting process $N(\cdot)$ corresponding to no dead-time (uncensored process). This is the advantage of the formula (2.5).

† From now on we set $\alpha = 1$.

To obtain an expression for $f_a(\cdot)$, we note it is more convenient to deal with $F_a(\cdot)$ where

$$F_a(x)=\int_x^{\infty} f_a(y)\,\mathrm{d}y \tag{2.6}$$

for $F_a(x)$ is the stationary probability that no photocount is recorded by the detector up to the time point t_0+x given that a photocount is recorded in an arbitarily small time interval preceding t_0. From (2.5) and (2.6) we find that the function $F_a(\cdot)$ is given by

$$F_a(x)=\frac{\langle I(t_0)\exp(-\int_{t_0+a}^{t_0+x} I(u)\,\mathrm{d}u)\rangle}{\bar{I}}. \tag{2.7}$$

The right-hand side of (2.7) can be computed easily if we are in possession of the double generating function $G_2(s_1, t_1, T_1; s_2, t_2, T_2)$ of the photocounts defined by

$$G_2(s_1, t_1, T_1; s_2, t_2, T_2)=\left\langle \exp\left[-\left(s_1\int_{t_1}^{T_1} I(t)\,\mathrm{d}t+s_2\int_{t_2}^{T_2} I(t)\,\mathrm{d}t\right)\right]\right\rangle. \tag{2.8}$$

It is easy to note that

$$F_a(x)\bar{I}=\left.\frac{\partial^2 G_2}{\partial s_1\,\partial t_1}\right|_{\substack{s_1=0, s_2=1, t_1=t_0, T_1=t+a,\\ t_2=t_0+a, T_2=t_0+x}}. \tag{2.9}$$

An explicit expression for G_2 is available in the literature for some specific spectral profiles (see Jakeman 1970, Srinivasan *et al* 1973). If we confine ourselves to a Lorentzian profile of half-width Γ, G_2 is given by (Srinivasan *et al* 1973)

$$G_2(s_1, t_1, T_1; s_2, t_2, T_2)=1/A \tag{2.10}$$

where

$$A=g_1(T_1-t_1)g_2(T_2-t_2)[\exp(-\Gamma(T_1-t_1+T_2-t_2))]-\frac{1}{4}\left(\frac{\Gamma}{p_1}-\frac{p_1}{\Gamma}\right)\left(\frac{\Gamma}{p_2}-\frac{p_2}{\Gamma}\right)$$

$$\times\sinh[\bar{p}_1(T_1-t_1)]\sinh[\bar{p}_2(T_2-t_2)]\exp(-\Gamma(t_2-t_1+T_2-T_1)) \tag{2.11}$$

$$g_i(t)=\cosh(\bar{p}_i t)+\tfrac{1}{2}[(\Gamma/\bar{p}_i)+(\bar{p}_i/\Gamma)] \tag{2.12}$$

$$\bar{p}_i=(\Gamma^2+2\Gamma I s_i)^{1/2}. \tag{2.13}$$

Performing the differentiation as indicated in (2.9), we finally obtain

$$F_a(x)=\frac{1}{A(x)}-\frac{\bar{I}}{p}\,\mathrm{e}^{-\Gamma(x+a)}\frac{\sinh[p(x-a)]}{(A(x))^2} \qquad x>a \tag{2.14}$$

where $A(x)$ is the value of A (as given by (2.11)) evaluated at $s_1=0$, $s_2=-1$, $t_1=t_0$, $T_1=t+a=t_2$, $T_2=t_0+x$ and is given by

$$A(x)=\{\cosh[p(x-a)]+\tfrac{1}{2}[(p/\Gamma)+(\Gamma/p)]\sinh[p(x-a)]\}\,\mathrm{e}^{-\Gamma(x-a)} \tag{2.15}$$

$$p=(\Gamma^2+2\Gamma\bar{I})^{1/2}. \tag{2.16}$$

We note that

$$F_a(x)=1 \qquad \text{for } x<a. \tag{2.17}$$

Using (2.14), we obtain after integration†

$$\frac{1}{\beta\bar{I}} = a + b\sum_{n=0}^{\infty}\frac{c^n}{(2n+1)p-\Gamma}\left(1-\frac{Ib(n+1)\,\mathrm{e}^{-2\Gamma a}}{(2n+2)p-\Gamma}\right) \tag{2.18}$$

where

$$b = 4\Gamma p/(\Gamma+p)^2 \qquad c = (p-\Gamma)^2/(p+\Gamma)^2. \tag{2.19}$$

A consistency check can be provided by setting $a = 0$. In that case β must be equal to unity. The verification is best done at the level of the distribution function $F_a(x)$ as given by (2.14). Integrating the right-hand side of (2.10) with respect to x over the range 0 to ∞, we obtain

$$\int_0^{\infty} F_0(x)\,\mathrm{d}x = 1/\bar{I} \tag{2.20}$$

so that β equals to unity. Incidentally from (2.18) we obtain the result

$$I/\bar{I} = b\sum_{n=0}^{\infty}\frac{c^n}{(2n+1)p-\Gamma}\left(1-\frac{\bar{I}b(n+1)}{(2n+3)p-\Gamma}\right) \tag{2.21}$$

Using (2.16) we obtain

$$\beta = \left(1 + a\bar{I} + b^2\bar{I}^2\sum_{n=0}^{\infty}\frac{(n+1)c^n(1-\mathrm{e}^{-2\Gamma a})}{[(2n+1)p-\Gamma][(2n+3)p-\Gamma]}\right)^{-1}. \tag{2.22}$$

Next we consider the general case when the dead-time is governed by a probability density function $q(\cdot)$. We will assume that the different times are independently and identically distributed with the common probability density function $q(\cdot)$. In that case the survivor function $F_a(x)$ given by (2.14) holds good and can be interpreted as follows:

$F_a(x) = \Pr\{$the stationary time interval between two successive counts exceeds $x|$ the dead-time $= a\}$ (2.23)

Thus $F(x)$ the probability that the interval exceeds x is given by

$$F(x) = \int_0^{x} F(x)q(a)\,\mathrm{d}a + \int_x^{\infty} q(a)\,\mathrm{d}x. \tag{2.24}$$

On the other hand the expected value of the time interval between two counts is given by

$$\begin{aligned} 1/\beta\bar{I} &= \int_0^{\infty} F(x)\,\mathrm{d}x \\ &= \bar{a} + b\sum_{n=0}^{\infty}\frac{c^n}{(2n+1)p-\Gamma}\left(1-\frac{\bar{I}b(n+1)}{(2n+3)p-\Gamma}q^*(2\Gamma)\right) \end{aligned} \tag{2.25}$$

where $q^*(2\Gamma)$ is the Laplace Transform of $q(\cdot)$ evaluated at 2Γ. Using (2.21) we find

$$\beta = \left(1 + \bar{a}\bar{I} + b^2\bar{I}^2(1-q^*(2\Gamma)\sum_{n=0}^{\infty}\frac{c^n(n+1)}{[(2n+1)p-\Gamma][(2n+2)p-\Gamma]}\right)^{-1}. \tag{2.26}$$

The above result giving the stationary value of β is perhaps the best result that is possible. The information that can be obtained from (2.24) is very limited indeed.

† The expression given in Srinivasan (1975) is incorrect and it should be replaced by (2.18).

Second-order interval characteristics may be obtained by the procedure outlined above; however the computation is tedious since we have to deal with G_4, the four-fold generating function of the counts. Although the general method of obtaining G is straightforward (see Srinivasan *et al* 1973) the computation of the appropriate derivative is a fairly messy job. We will not pursue this since even this will throw light only on the stationary correlation between the successive intervals.

3. A lower bound for the mean square number

As mentioned in an earlier communication (Srinivasan 1975) the most valuable information on the statistics is given by $h(\cdot)$ the stationary conditioned product density of degree one of detected photons. Since there appears to be no possibility of obtaining an explicit expression for $h(\cdot)$ in the near future, it may be worthwhile to obtain a lower bound for the mean square number of detected photons in any arbitrary interval.

The lower bound is naturally provided by the counting process obtained by censoring the process (see Ramakrishnan and Mathews 1953, Smith 1958) through a type-II counter. In this arrangement each event (photon) whether registered or not gives rise to a dead-time a which we shall assume to be fixed. In theory a counter of this type can be indefinitely locked in the sense that no photon can be detected. Obviously the average number of photocounts recorded by the detector is smaller than $\beta\bar{I}$. The same statement can be made for the mean square number of photocounts and hence this provides a lower bound. Defining $h_1^c(\cdot)$ the stationary conditioned product density of degree one of the censored process by

$$h_1^c(x) = \lim_{\Delta,\Delta'\to 0} \Pr\{N_a^c(t_0+x+\Delta) - N_a^c(t_0+x) = 1 | N_a^c(t_0) - N_a^c(t_0-\Delta') = 1\}/\Delta \tag{3.1}$$

where $N_a^c(\cdot)$ is the counting process obtained by censoring the photocounts process through a type-II counter, we obtain

$$h_1^c(x) = \begin{cases} \dfrac{\langle \beta_c I(t_0)[\exp(-\int_{t_0+x-a}^{t_0+x} I(u)\,du)]I(t_0+x)\rangle}{\langle \beta_c I(t_0)\rangle}, & x > 2a \\[2ex] \dfrac{\langle \beta_c I(t_0)[\exp(-\int_{t_0+a}^{t_0+x} I(u)\,du)]I(t_0+x)\rangle}{\langle \beta_c I(t_0)\rangle}, & a < x < 2a \\[2ex] 0, & 0 < x < a \end{cases} \tag{3.2}$$

where β_c is the stationary value of $\beta_c(t)$ the probability that the counter is open at t. Equation (3.2) is obtained by using arguments similar to those used for the derivation of (2.5) with the only difference being that the photocount at t_0+x is not necessarily the first one counted from t_0. Moreover the counter is of type II and as such attention need be devoted only to the interval of length a preceding t_0+x. The rest of the arguments run on parallel lines. Next we notice that $h_1^c(\cdot)$ is related to $G_2(s_1, t_1, T_1; s_2, t_2, T_2)$ (see (2.8)) by

$$h_1^c(x) = -\frac{1}{\bar{I}} \frac{\partial^3 G}{\partial T_2\, \partial t_1\, \partial s_1} \tag{3.3}$$

at the point $s_1 = 0$, $s_2 = 1$, $t_1 = t_0$, $t_2 = t_0 + x - a$, $T_1 = t_0 + a$, $T_2 = t_0 + x$ if $x > 2a$. On the other hand if $a < x < 2a$, the derivative is evaluated at the point $s_1 = 0$, $s_2 = 1$, $t_1 = t_0$, $t_2 = T_1 = t_0 + a$, $T_2 = t_0 + x$. It can be proved that $h(\cdot)$, the corresponding product density of the detected photoelectrons will be greater in magnitude than $h_1^c(\cdot)$ point wise. Thus $h_1^c(\cdot)$ provides a lower bound for the corresponding product density $h(\cdot)$. Using the formula for moments (Srinivasan 1974b, chap. 4), we have†

$$\langle (N_a^c(t_0+T)-N_a^c(T))^2\rangle = \langle N_a^c(t_0+T)-N_a^c(T)\rangle + 2h_1^c(\infty)\int_0^T (T-x)h_1^c(x)\,\mathrm{d}x. \qquad (3.4)$$

We can obtain an improved approximation to the bound by replacing the first term by the expected value of the detected photoelectrons. A further improvement is obtained by noting $h_1^c(\infty)$ is the rate of counts and can be replaced by the actual rate which is $\beta\bar{I}$. Thus we have

$$\langle [N_a(t_0+T)-N_a(t_0)]^2\rangle_{\text{lower bound}} = \beta\bar{I}T + 2\bar{I}\int_0^T (T-x)h_1^c(x)\,\mathrm{d}x. \qquad (3.5)$$

Thus the evaluation of the bound reduces to that of finding an expression for $h_1^c(\cdot)$. Using (3.3) and (2.10) through (2.12) we obtain after a lengthy but straightforward computation

$$h_1^c(x) = \begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{A(x)} + \dfrac{1}{2(A(x))^2}\left(\dfrac{\Gamma}{p}-\dfrac{p}{\Gamma}\right)\mathrm{e}^{-\Gamma(x+a)}\sinh[p(x-a)]\right], & a<x<2a \\ h_1^c(\infty) + \mathrm{e}^{-2\Gamma(x-a)}L(a), & x>2a \end{cases} \qquad (3.6)$$

where

$$L(a) = \bar{I}\,\mathrm{e}^{-\Gamma a}\{4\,\mathrm{e}^{-\Gamma a}\sinh(pa)[p\sinh(pa) + (\Gamma+\bar{I})\cosh(pa)] + A(2a)[p\cosh(pa)+\sinh(pa)]\}(A(2a))^{-2} \qquad (3.7)$$

$$h_1^c(\infty) = \bar{I}\,\mathrm{e}^{-\Gamma a}[\cosh(pa) - (\Gamma/p)\sinh(pa)](A(2a))^{-2} \qquad (3.8)$$

and $A(\cdot)$ is still given by (2.15).

No further attempt will be made to compute the integral on the right-hand side of (3.5) since it is straightforward. A still further refinement to the bound is achieved by replacing $h_1^c(\infty)$ on the right-hand side of (3.6) by $\beta\bar{I}$. Finally we note that it is a difficult task to extend formula (3.6) to cover the case when a has a distribution. We must be content with the weak result relating to the rate of counts provided in § 2. The problem will perhaps remain open for quite some time.

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† For want of a better notation we use the symbol $N_a(\cdot)$ to denote the *physical* counting process of photoelectrons counted by the detector.

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